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On Certain Classes of Meromorphic Functions

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ABSTRACT:
A meromorphic function with a simple pole at $z = 0$ and of the form $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ for $z \in D \equiv \{z \in \mathbb{C}: 0 < |z| < 1\}$ with $f(z) \neq 0$ in D can be expressed as $f(z) = \frac{1}{zg(z)}$ where $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ in D. In this paper certain coefficient criteria are derived for some classes of meromorphic functions.

KEYWORDS: Meromorphic function, univalent starlike function.

INTRODUCTION

Let $\widetilde{M}$ denote the class of functions which are analytic in $D = D(1)$ where
$D(r) = \{z \in \mathbb{C}: 0 < |z| < r\}$ for $r > 0$,
with a simple pole at the point $z = 0$ and $\mathbb{C}$ being the set of complex numbers. By M, we denote the class of functions $f \in \widetilde{M}$ of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \ (z \in D) \quad (1)$$

Also, by τ_η^ε $(\eta \in R,\ \varepsilon \in \{0,1\})$, we denote the class of functions $f \in M$ of the form (1) for which

$$\arg(a_n) = \varepsilon\pi - (n+1)\eta \quad (n \in N \equiv \{1,2,3,\ldots\}).$$

For $\eta = 0,\ \varepsilon = 0$ we obtain the class τ_0^0 of functions with positive coefficients.
Motivated by Silverman[3], Djiok[1] defined the class

$$\tau^\varepsilon \equiv \bigcup_{\eta \in \mathbf{R}} \tau_\eta^\varepsilon .$$

It is called the class of functions with varying coefficients.
Let $\alpha \in (0,1)$, $r \in (0,1)$. A function $f \in M$ is said to be meromorphically starlike of order α in $D(r)$ if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) < -\alpha \quad (z \in D(r)). (2)$$

Djiok[1] introduced the class of all functions in M, which are meromorphically starlike of order α and denoted it by $MS^*(\alpha)$.
We set $MS^* = MS^*(0)$.
For a function $f \in \tau_\eta^0$, the condition (2) is equivalent to

$$\left|\frac{zf'(z)}{f(z)} + 1\right| < 1 - \alpha \quad (z \in D(r)). \quad (3)$$

Let us define a new class $MS^*(A,B)$ which generalizes $S^*(\alpha)$:
A function $f \in M$ is said to be in the class $MS^*(A,B)$ if

$$-\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} (z \in D(r)),$$

where $-1 \leq B < A \leq 1$ and "$\phi \prec \mu$" means that $\phi(D) \subseteq \mu(D)$. We have

$$MS^*(1-2\alpha, -1) = MS^*(\alpha).$$

Kulkarni and Joshi[2] studied the class $\sum(\alpha,\beta,\gamma)$ of functions $f \in \Sigma$ satisfying the condition

$$\left|\frac{\frac{zf'(z)}{f(z)}+1}{2\gamma\left(\frac{zf'(z)}{f(z)}+\alpha\right) - \left(\frac{zf'(z)}{f(z)}+1\right)}\right| \leq \beta (4)$$

for

$$(z \in D)\left(0 \leq \alpha < 1;\ 0 < \beta \leq 1;\ \frac{1}{2} < \gamma \leq 1\right).$$

Σ is the class of functions in $\widetilde{M}$ which are univalent in D.
In this paper we find sufficient conditions in terms of b_n's in Theorems-1, 2 and 3 for some subclasses of $MS^*(\alpha)$, $MS^*(A,B)$ and $\sum(\alpha,\beta,\gamma)$ respectively.

SECTION-1

In this section we find a sufficient condition in Theorem-1 for the subclass of $\tau_\eta^0 \cap MS^*(\alpha)$.

Theorem-1:
If $f(z) = \frac{1}{zg(z)} = \frac{1}{z\,(1+\sum_{n=1}^{\infty} b_n z^n)} \in \tau_\eta^0$
for $D = \{z: 0 < |z| < 1\}$ with $f(z) \neq 0$ in D, $0 < \alpha < 1$ and $b_n \in \mathbb{C}$ for $n \in N$ satisfy
$\sum_{n=1}^{\infty}[\{n + (1-\alpha)\}|b_n|] < 1 - \alpha$ (5)
then $f \in \tau_\eta^0 \cap MS^*(\alpha)$.

Proof:
Let $f(z) = \frac{1}{zg(z)}$ where $g(z) = \sum_{n=0}^{\infty} b_n z^n$
with $b_0 = 1$ and $0 < |z| < 1$.
We have

$$f'(z) = -\frac{g(z) + z\,g'(z)}{\{zg(z)\}^2}$$

$$\frac{zf'(z)}{f(z)} = -1 - \frac{zg'(z)}{g(z)}$$

$$\left|\frac{zf'(z)}{f(z)} + 1\right| = \left|-\frac{z\sum_{n=1}^{\infty} nb_n z^{n-1}}{\sum_{n=0}^{\infty} b_n z^n}\right|$$

$$\leq \frac{\sum_{n=1}^{\infty} n|b_n|}{1 - \sum_{n=1}^{\infty}|b_n|}$$

$$< 1 - \alpha$$

by the given condition (3).

Now the inequality (5) gives the required conclusion of the theorem.

Next we find a sufficient condition in Theorem-2 for the subclass of $\tau_\eta^0 \cap MS^*(A, B)$.

Theorem-2:
If $f(z) = \frac{1}{zg(z)} = \frac{1}{z\,(1+\sum_{n=1}^{\infty} b_n z^n)} \in \tau_\eta^0$
for $D = \{z: 0 < |z| < 1\}$with $f(z) \neq 0$ in D, $-1 \leq B < A \leq 1$ and $b_n \in \mathbb{C}$ for $n \in N$ satisfy
$\sum_{n=1}^{\infty}[\{n + n|B| + (A-B)\}|b_n|] < A - B$(6)
then $f \in \tau_\eta^0 \cap MS^*(A, B)$.

Proof:
Let $f(z) = \frac{1}{zg(z)}$ where $g(z) = \sum_{n=0}^{\infty} b_n z^n$
with $b_0 = 1$ and $0 < |z| < 1$.
We have

$$f'(z) = -\frac{g(z) + z\,g'(z)}{\{zg(z)\}^2}$$

$$-\frac{zf'(z)}{f(z)} = 1 + \frac{zg'(z)}{g(z)}$$

$$= \frac{1+Aw(z)}{1+Bw(z)} \quad \text{say.}$$

We get

$$w(z) = -\frac{zg'(z)}{(B-A)g(z) + Bzg'(z)}$$

$$= \frac{-\sum_{n=1}^{\infty} n\,b_n z^n}{(B-A)\sum_{n=0}^{\infty} b_n z^n + B\sum_{n=1}^{\infty} n\,b_n z^n}$$

Hence

$$|w(z)| \leq \frac{\sum_{n=1}^{\infty} n\,|\,b_n|}{|B-A| - \sum_{n=1}^{\infty}(A - B + n|B|)|b_n|}$$

Now, this and the given condition (6) give that $|w(z)| \leq 1$.
Further $w(0) = 0$. Thus
$-\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}$.
Hence $f \in \tau_\eta^0 \cap MS^*(A, B)$.

Corollary: For $A = 1 - 2\alpha$, $B = -1$, Theorem-2 gives Theorem-1.

SECTION-2

Here we find a sufficient condition in Theorem 3 for the class $\sum(\alpha, \beta, \gamma)$

Theorem-3:
If $f(z) = \frac{1}{zg(z)} = \frac{1}{z\,(1+\sum_{n=1}^{\infty} b_n z^n)}$ for $z \in D$ is in Σ and $f(z) \neq 0$ in D with $b_n \in \mathbb{C}$ for $n \in N$ and $\left(0 \leq \alpha < 1;\ 0 < \beta \leq 1;\ \frac{1}{2} < \gamma \leq 1\right)$ satisfy
$\sum_{n=1}^{\infty}[n(1 + \beta(2\gamma - 1) + 2\gamma\beta(1-\alpha))]|b_n| < 2\beta\gamma(1-\alpha)$ (7)
then $f \in \sum(\alpha, \beta, \gamma)$.

Proof:
Let $f(z) = \frac{1}{zg(z)}$ where $g(z) = \sum_{n=0}^{\infty} b_n z^n$ for $0 < |z| < 1$ with $b_0 = 1$.
We have
$-\frac{zf'(z)}{f(z)} = = 1 + \frac{zg'(z)}{g(z)}$ for $z \in D$.
Further

$$\left|\frac{\frac{zf'(z)}{f(z)} + 1}{2\gamma\left(\frac{zf'(z)}{f(z)} + \alpha\right) - \left(\frac{zf'(z)}{f(z)} + 1\right)}\right|$$

$$= \left|\frac{-\frac{zg'(z)}{g(z)}}{2\gamma\left(-1 - \frac{zg'(z)}{g(z)} + \alpha\right) + \frac{zg'(z)}{g(z)}}\right|$$

$$= \left|\frac{zg'(z)}{2\gamma[(\alpha-1)g(z) - zg'(z)] + zg'(z)}\right|$$

$$= \left|\frac{\sum_{n=1}^{\infty} nb_n z^n}{2\gamma[(\alpha-1)\sum_{n=0}^{\infty} b_n z^n - \sum_{n=1}^{\infty} nb_n z^n] + \sum_{n=1}^{\infty} nb_n z^n}\right|$$

$$= \frac{\sum_{n=1}^{\infty} n|b_n|}{|\sum_{n=0}^{\infty}[2\gamma\{(\alpha-1) - n\} + n]\,b_n z^n|}$$

$$\leq \frac{\sum_{n=1}^{\infty} n|b_n|}{2\gamma(1-\alpha) - \sum_{n=1}^{\infty}|2\gamma\{(\alpha-1) - n\} + n||b_n|}$$

$$= \frac{\sum_{n=1}^{\infty} n|b_n|}{2\gamma(1-\alpha) - \sum_{n=1}^{\infty}\{2\gamma(1-\alpha)+n(2\gamma-1)\}|b_n|}$$

$\leq \beta$ by (7) (8)

Thus (4) and (8) give that

$f \in \sum(\alpha, \beta, \gamma)$.

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